

Existence of the Solution for the 't Hooft-Polyakov Monopole

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Abstract

In this paper we give a mathematical proof of the existence of the time independent and spherically symmetric solution to the 't Hooft-Polyakov model of magnetic monopole by using 2D-shooting method.

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1. Introduction

The existence problem of the magnetic monopole has fastinated physicists since Dirac's classic work [1] before fifty years ago. In 1974, 't Hooft [2] and Polyakov [3] proposed a model for a magnetic monopole which arises as a static solution of the classical equations for the $SU(2)$ Yang-Mills field coupled to an $SU(2)$ Higgs field. The model has been extended by Julia and Zee [4], and Cho and Maison [5]. In this paper, we discuss the existence and asymptotics for the solution of 't Hooft-Polyakov monopole. Prasad and Sommerfield [6] found exact solutions of the field equation for a special case ($\lambda = 0$). But so far to our knowledge the boundary value problem for the equation of motion with $\lambda > 0$ was just studied numerically. And the existence of the 't Hooft-Polyakov monopole is just convinced by numerical computations. In this paper we give a mathematical proof for the existence of the 't Hooft-Polyakov magnetic monopole.

The Lagrangian of the 't Hooft-Polyakov model [2] [3] is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a - \frac{1}{2}D_\mu\phi_a D_\mu\phi_a + \frac{1}{2}\mu^2\phi_a\phi_a - \frac{1}{4}\lambda(\phi_a\phi_a)^2, \quad (1)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\epsilon_{abc}A_\mu^b A_\nu^c, \quad (2)$$

$$D_\mu\phi_a = \partial_\mu\phi_a + e\epsilon_{abc}A_\mu^b\phi_c. \quad (3)$$

The Wu-Yang^[7]-'t Hooft-Polyakov Ansatz is to seek a solution of the equations of motion (which can be derived from the Lagrangian, see for example [6]) in the form

$$A_i^a = \epsilon_{aij}\frac{x_j}{er^2}(1 - K(r)), \quad (4)$$

$$A_0^a = 0, \quad (5)$$

$$\phi_a = \frac{x_a}{er^2}H(r). \quad (6)$$

where $K(r), H(r)$ satisfy the equations

$$r^2 K'' = K(K^2 + H^2 - 1), \quad (7)$$

$$r^2 H'' = 2HK^2 + \frac{\lambda}{e^2}(H^3 - e^2\rho_0^2 r^2 H), \quad (8)$$

where $\rho_0 = \mu/\sqrt{\lambda}$. Let's put

$$e = g_0, K(r) = f(r), H(r) = g_0 r \rho(r), \quad (9)$$

then the equations (7), (8) reduce to

$$f'' - \frac{f^2 - 1}{r^2} f = g_0^2 \rho^2 f, \quad (10)$$

$$\rho'' + \frac{2}{r} \rho' - \frac{2f^2}{r^2} \rho = \lambda(\rho^2 - \rho_0^2) \rho, \quad (11)$$

with the boundary conditions [5]

$$\begin{aligned} f(0) &= 1, & \rho(0) &= 0, \\ f(\infty) &= 0, & \rho(\infty) &= \rho_0. \end{aligned} \quad (12)$$

Because the exact solution was already found for $\lambda = 0$ [6], we just discuss the problem for $\lambda > 0$ in this paper. To solve this boundary value problem, we consider the asymptotics as $r \rightarrow 0$,

$$f(r) \sim 1 - \alpha r^2, \quad (13)$$

$$\rho(r) \sim \beta r, \quad (14)$$

where $\alpha, \beta > 0$. We are going to use topological 2D-shooting method [8], [9] to prove that there exist such α, β such that the corresponding f, ρ satisfy $f(\infty) = 0, \rho(\infty) = \rho_0$. That is we want to prove the following theorem in this paper.

Theorem 1 *For $\lambda > 0, \rho_0 > 0, g_0 > 0$, there is a solution (f, ρ) to equations (10), (11) satisfying the boundary conditions (12), and*

$$0 < f < 1, \quad 0 < \rho < \rho_0, \quad (15)$$

$$f' < 0, \quad \rho' > 0, \quad (16)$$

for $0 < r < \infty$.

The plan of this paper is as follows. In sec.2, we consider the asymptotic behaviour of the solution to the eqs. (10), (11). Then we discuss the behaviours of $f(r)$ as α small or large for β in a finite interval. We show that f' cross 0 when α small in sec. 3, and f cross 0 when α large in sec. 4. In sec. 5 we talk about the possibilities of the behaviours for ρ when f stays between 0 and 1. While f stays between 0 and 1, we show that when β is small ρ crosses 0 in sec. 6, and as β large ρ crosses ρ_0 in sec. 7. Finally by the topological lemma (McLeod and Serrin [8]) we show that the solution to the boundary value problem (10), (11) and (12) exists.

2. Asymptotics of Solution at the Origin

Lemma 1 *For any α, β , there is a unique solution (f, ρ) to the equations (10), (11), such that*

$$f \sim 1 - \alpha r^2, \quad (1)$$

$$\rho \sim \beta r, \quad (2)$$

as $r \rightarrow 0$.

Proof Set

$$s = \log r, f(r) = 1 + p(s), \rho(r) = q(s). \quad (3)$$

Then eqs. (10), (11) are reduced to

$$p'' - p' - 2p = 3p^2 + p^3 + g_0^2 e^{2s} q^2 p, \quad (4)$$

$$q'' + q' - 2q = 2(2p + p^2)q + \lambda e^{2s} (q^2 - \rho_0^2)q, \quad (5)$$

which is at least formally equivalent to the integral equations

$$p(s) = -\alpha e^{2s} + \frac{1}{3} \int_{-\infty}^s (e^{2(s-\sigma)} - e^{-(s-\sigma)}) (3p^2 + p^3 + g_0^2 e^{2\sigma} q^2 p) d\sigma, \quad (6)$$

$$q(s) = \beta e^s + \frac{1}{2} \int_{-\infty}^s (e^{s-\sigma} - e^{-2(s-\sigma)}) (2(2p + p^2)q + \lambda e^{2\sigma} (q^2 - \rho_0^2)q) d\sigma. \quad (7)$$

Let

$$p = e^{2s} \phi, q = e^s \psi.$$

The eqs. (6) and (7) become

$$\phi = -\alpha + \frac{1}{3} \int_{-\infty}^s (e^{2\sigma} - e^{-3s+5\sigma}) (3\phi^2 + e^{2\sigma} \phi^3 + g_0^2 e^{2\sigma} \psi^2 \phi) d\sigma, \quad (8)$$

$$\psi = \beta + \frac{1}{2} \int_{-\infty}^s (e^{2\sigma} - e^{-3s+5\sigma}) (2(2\phi + e^{2\sigma} \phi^2) \psi + \lambda (e^{2\sigma} \psi^2 - \rho_0^2) \psi) d\sigma, \quad (9)$$

which can be solved by iteration by setting

$$\phi_0 = -\alpha, \psi_0 = \beta,$$

$$\phi_{n+1}(s) = -\alpha + \frac{1}{3} \int_{-\infty}^s (e^{2\sigma} - e^{-3s+5\sigma}) (3\phi_n^2 + e^{2\sigma} \phi_n^3 + g_0^2 e^{2\sigma} \psi_n^2 \phi_n) d\sigma,$$

$$\psi_{n+1}(s) = \beta + \frac{1}{2} \int_{-\infty}^s (e^{2\sigma} - e^{-3s+5\sigma}) (2(2\phi_n + e^{2\sigma} \phi_n^2) \psi_n + \lambda (e^{2\sigma} \psi_n^2 - \rho_0^2) \psi_n) d\sigma,$$

for $n \geq 0$.

We choose constant numbers K, S by

$$K = \max(2|\alpha|, 2|\beta|, 3), \quad (10)$$

$$e^{2S} = \max(M_1, M_2, M_3, M_4), \quad (11)$$

where

$$M_1 = \frac{1}{2}(2(2K + K^2) + \lambda(K^2 + \rho_0^2)),$$

$$M_2 = \frac{1}{3}(6K + 2K^2 + 2g_0^2K^2),$$

$$M_3 = (8 + 3\lambda)K^2 + \lambda\rho_0^2,$$

$$M_4 = 4K + 3(1 + g_0^2)K^2.$$

We claim that for $s \in (-\infty, -S]$, there are

$$|\phi_n|, |\psi_n| \leq K, \quad (12)$$

$$|\phi_{n+1} - \phi_n| \leq \frac{M}{3^{n+1}}e^{2s}, \quad (13)$$

$$|\psi_{n+1} - \psi_n| \leq \frac{M}{3^{n+1}}e^{2s}, \quad (14)$$

for $n \geq 0$, where

$$M = \max(4K^3 + \lambda(K^2 + \rho_0^2)K, (2 + g_0^2)K^3). \quad (15)$$

This can be proved by induction. We skip the details here. Hence $\{(\phi_n, \psi_n)\}_{n=0}^\infty$ is convergent. The uniqueness is similar.

3. The Solutions for Small α

Lemma 2 *For any $\beta_2 \geq \beta_1 > 0$, when $\beta \in [\beta_1, \beta_2]$, there exists $\alpha_1 > 0$, such that if $\alpha \in (0, \alpha_1]$, there is $r^- > 0$, so that*

$$f'(r^-) > 0, \quad (1)$$

$$f(r) > 0, \text{ for } 0 < r \leq r^-. \quad (2)$$

Proof We make a scaling

$$t = \frac{r}{\sqrt{\alpha}}, \quad f(r) = 1 + \alpha^2 p(t), \quad \rho(r) = \sqrt{\alpha} q(t). \quad (3)$$

Then the equations become

$$p'' - \frac{2p + \alpha^2 p^2}{t^2} (1 + \alpha^2 p) = g_0^2 q^2 (1 + \alpha^2 p), \quad (4)$$

$$q'' + \frac{2}{t} q' - \frac{2(1 + \alpha^2 p)^2}{t^2} q = \lambda \alpha (\alpha q^2 - \rho_0^2) q, \quad (5)$$

with the asymptotics at the origin

$$p(t) \sim -t^2, \quad (6)$$

$$q(t) \sim \beta t, \quad (7)$$

as $t \rightarrow 0$.

Letting $\alpha = 0$, the problem is reduced to

$$P(t) \sim -t^2, \quad Q(t) \sim \beta t, \quad t \rightarrow 0, \quad (8)$$

$$P'' - \frac{2P}{t^2} = g_0^2 Q^2, \quad (9)$$

$$Q'' + \frac{2}{t} Q' - \frac{2}{t^2} Q = 0. \quad (10)$$

There is a unique solution for this problem. It's not difficult to see that the solution is

$$P(t) = -t^2 + \frac{g_0^2 \beta^2}{10} t^4, \quad (11)$$

$$Q(t) = \beta t, \quad (12)$$

and then

$$P'(t) = 2t(-1 + \frac{g_0^2 \beta^2}{5} t^2). \quad (13)$$

For $\beta \in [\beta_1, \beta_2]$, choose small $\epsilon^- > 0$, and

$$t_0 = \frac{\sqrt{5}}{g_0 \beta_1} + \epsilon^-, \quad (14)$$

such that

$$P'(t_0) > 0, \quad (15)$$

$$|P(t)| \leq t_0^2 + \frac{g_0^2 \beta_2^2}{10} t_0^4, 0 \leq t \leq t_0. \quad (16)$$

Now for the solution p, q to the equations (4), (5) with the conditions (6) and (7), since p, p', q are continuous in r, α, β , by uniform continuity in compact set, there exists $\alpha_1 > 0$ satisfying

$$2\alpha_1^2 < (t_0^2 + \frac{g_0^2 \beta_2^2}{10} t_0^4)^{-1},$$

such that if $\alpha \in (0, \alpha_1], \beta \in [\beta_1, \beta_2]$,

$$p'(t_0) > 0, \quad (17)$$

$$|p(t)| \leq 2 \left(t_0^2 + \frac{g_0^2 \beta_2^2}{10} t_0^4 \right), 0 \leq t \leq t_0, \quad (18)$$

which implies that for $\alpha \in (0, \alpha_1]$

$$\begin{aligned} f'(r^-) &= \alpha^{3/2} p' \left(\frac{r^-}{\sqrt{\alpha}} \right) > 0, \\ f(r) &> 1 - \alpha^2 |p(t)| \\ &\geq 1 - 2\alpha_1^2 \left(t_0^2 + \frac{g_0^2 \beta_2^2}{10} t_0^4 \right) > 0, \quad 0 < r \leq r^-, \end{aligned}$$

where $r^- = \sqrt{\alpha} t_0$. So the lemma is proved.

4. The Solutions for Large α

Lemma 3 *For any $\beta_2 \geq \beta_1 \geq 0$, when $\beta \in [\beta_1, \beta_2]$, there is a large $\alpha_2 > 0$, such that if $\alpha \in [\alpha_2, \infty)$, there exists $r^+ > 0$, so that*

$$f(r^+) < 0, \quad (1)$$

$$f'(r) < 0, 0 < r \leq r^+. \quad (2)$$

Proof We make another scaling

$$t = \sqrt{\alpha} r, f(r) = 1 - \psi(t), \rho(r) = \phi(t). \quad (3)$$

Then the equations become

$$\psi'' - \frac{\psi(1-\psi)(2-\psi)}{t^2} = \frac{-1}{\alpha} g_0^2 \phi^2 (1-\psi), \quad (4)$$

$$\phi'' + \frac{2}{t} \phi' - \frac{2(1-\psi)^2}{t^2} \phi = \frac{1}{\alpha} \lambda (\phi^2 - \rho_0^2) \phi, \quad (5)$$

with the asymptotics as $t \rightarrow 0$

$$\begin{aligned} \psi(t) &\sim t^2, \\ \phi(t) &\sim \frac{1}{\sqrt{\alpha}} \beta t. \end{aligned}$$

Then as $\alpha \rightarrow \infty$, $\phi \rightarrow 0$ uniformly on compact intervals in t , while ψ tends, also uniformly on compact intervals in t , to the solution Ψ of

$$\begin{aligned} \Psi'' &= \frac{\Psi(1-\Psi)(2-\Psi)}{t^2}, \\ \Psi(t) &\sim t^2, t \rightarrow 0. \end{aligned}$$

To get the behaviour of Ψ , we make a transformation

$$s = \log t.$$

Then the equation is reduced to

$$\begin{aligned} \Psi_{ss} &= \Psi_s + 2\Psi - 3\Psi^2 + \Psi^3, -\infty < s < \infty, \\ \Psi(s) &\sim e^{2s}, s \rightarrow -\infty. \end{aligned}$$

Multiplying this equation by $d\Psi/ds$ and integrating, we arrive at

$$\frac{1}{2} \Psi_s^2 = \Psi^2 \left(1 - \frac{\Psi}{2}\right)^2 + \int_{-\infty}^s \Psi_\sigma^2 d\sigma.$$

This makes clear that $d\Psi/ds$ does not vanish and Ψ becomes unbounded and certainly crosses 1, while $d\Psi/ds$ keeps positive at least before the crossing.

By the same argument as in lemma 1, we see that for $\beta \in [\beta_1, \beta_2]$, there is $\alpha_2 > 0$, such that if $\alpha \geq \alpha_2$, there exists $r^+ = r^+(\alpha)$, so that the lemma holds.

5. Argument for $\alpha \notin S_\beta^- \cup S_\beta^+$

For any $\beta > 0$, define

$$\begin{aligned} S_\beta^- &= \{\alpha > 0 \mid f' \text{ cross } 0 \text{ before } f \text{ cross } 0\}, \\ S_\beta^+ &= \{\alpha > 0 \mid f \text{ cross } 0 \text{ before } f' \text{ cross } 0\}. \end{aligned}$$

By Lemma 2,3, S_β^- and S_β^+ are not empty and disjoint. By implicit function theorem, it's not hard to prove that S_β^-, S_β^+ are open sets, so that $(0, \infty) \setminus (S_\beta^- \cup S_\beta^+)$ is not empty and closed set. For $\alpha \in (0, \infty) \setminus (S_\beta^- \cup S_\beta^+)$, we simply denote it as $\alpha \notin (S_\beta^- \cup S_\beta^+)$. By eq. (10) we see that if $f = 0, f' = 0$ at the same time, then the f is identically zero, which is impossible. So we have proved the following lemma.

Lemma 4 *If $\alpha \notin (S_\beta^- \cup S_\beta^+)$, then*

$$0 < f < 1, f' \leq 0,$$

for $0 < r < \infty$.

Lemma 5 *For $\beta > 0, \alpha \notin S_\beta^- \cup S_\beta^+$, there are three possibilities for ρ ,*

- (A) ρ' cross 0 at some point $r = r_0$, while $0 < \rho < \rho_0$, for $0 < r \leq r_0$.
- (B) ρ cross ρ_0 .
- (C) $0 < \rho < \rho_0, \rho' \geq 0$, for $0 < r < \infty$, and

$$\rho(\infty) = \rho_0, \tag{1}$$

$$f(\infty) = 0. \tag{2}$$

Proof Because $\rho'(0) = \beta > 0$, if ρ does not cross ρ_0 (case (B)), then either ρ' crosses 0 at some point $r = r_0$, while $0 < \rho < \rho_0$, for $0 < r \leq r_0$ (case (A)), or $\rho' \geq 0, 0 < \rho < \rho_0$, for $0 < r < \infty$ (case (C)). There is no possibility that when ρ does not cross ρ_0 , $\rho = \rho_0$ at some point. In fact, if $\rho = \rho_0$ and $\rho' = 0$ at the same time, then by eq. (11) we have $\rho'' > 0$ at this point, since $\beta > 0, \alpha \notin S_\beta^- \cup S_\beta^+$. Then ρ crosses ρ_0 , which is a contradiction.

For case (C), we have

$$f(0) = 1, 0 < f < 1, f' \leq 0, f'(\infty) = 0, f(\infty) = a, \tag{3}$$

$$\rho(0) = 0, 0 < \rho < \rho_0, \rho' \geq 0, \rho'(\infty) = 0, \rho(\infty) = b, \tag{4}$$

for some $a \in [0, 1)$, $b \in (0, \rho_0]$, where the second and third parts of (3), (4) are for $0 < r < \infty$. We want to show $a = 0, b = \rho_0$.

Suppose $b < \rho_0$. Choose $r_1 > 0$, so that $b/2 \leq \rho$, for $r_1 \leq r < \infty$. By eq. (11), there is

$$\begin{aligned} (r^2 \rho')' &= \rho + \lambda r^2 (\rho^2 - \rho_0^2) \rho \\ &\leq b - \lambda r^2 (\rho_0^2 - b^2) \frac{b}{2}, \end{aligned}$$

for $r_1 \leq r < \infty$. Integrting from r_1 to r , and dividing r^2 both sides, finally we get

$$\begin{aligned} \rho'(r) &\leq \frac{r_1^2}{r^2} \rho'(r_1) + b \left(\frac{1}{r} - \frac{r_1}{r^2} \right) - \frac{\lambda b}{6} (\rho_0^2 - b^2) \left(r - \frac{r_1^3}{r^3} \right) \\ &\rightarrow -\infty, \text{ as } r \rightarrow \infty. \end{aligned}$$

This is a contradiction. So $b = \rho_0$.

Now suppose $a > 0$. Choose $r_2 > 0$ so that

$$\begin{aligned} \frac{1 - a^2}{r^2} &\leq \frac{g_0^2 \rho_0^2}{8}, \\ \rho &\geq \rho_0/2, \end{aligned}$$

for $r_2 \leq r < \infty$. Then we have from eq. (10)

$$\begin{aligned} f'' &= f \left(\frac{f^2 - 1}{r^2} + g_0^2 \rho^2 \right) \\ &\geq f \left(\frac{a^2 - 1}{r^2} + \frac{g_0^2 \rho_0^2}{4} \right) \\ &\geq f \frac{g_0^2 \rho_0^2}{8} \\ &\geq \frac{a g_0^2 \rho_0^2}{8}. \end{aligned}$$

Integrating from r_2 to r , we get

$$f'(r) \geq f'(r_2) + \frac{a g_0^2 \rho_0^2}{8} (r - r_2) \rightarrow \infty, \text{ as } r \rightarrow \infty.$$

This is a contradictin. So $a = 0$. So the lemma is proved.

6. The Solutions for Small β

We make a transformation

$$t = \sqrt{\lambda}\rho_0 r, f(r) = p(t), \rho(r) = \beta q(t). \quad (1)$$

Then the equation and the asymptotics become

$$p'' - \frac{p^2 - 1}{t^2} p = \beta^2 g_1^2 q^2 p, \quad (2)$$

$$(P) \quad q'' + \frac{2}{t} q' + (1 - \frac{2p^2}{t^2}) q = \frac{\beta^2}{\rho_0^2} q^3, \quad (3)$$

$$p(t) \sim 1 - \frac{\alpha}{\lambda\rho_0^2} t^2, q(t) \sim \frac{t}{\sqrt{\lambda}\rho_0}, \text{ as } t \rightarrow 0, \quad (4)$$

where $g_1 = \lambda\rho_0^2$, and we will consider $\alpha \geq 0, \beta \geq 0$. The difference between this system and the system (10), (11) and (12) is only for the case $\beta = 0$, because when $\beta \neq 0$ these two systems are equivalent. But for $\beta = 0$ we can show by the same method that this problem has unique solution. For $\beta = 0, \alpha > 0$, by the same argument as in the proof of Lemma 3, we see that p cross 0. And for $\beta = 0, \alpha = 0$, there is $p = 1$.

Let's choose $\bar{\beta} > 0$. By Lemma 3 for $\beta \in [0, \bar{\beta}]$ there is $\bar{\alpha} > 0$, such that for any $\alpha \geq \bar{\alpha}$, we have $\alpha \in S_\beta^+$. Now let's define in the (α, β) plane the sets

$$D = [0, \bar{\alpha}] \times [0, \bar{\beta}], \quad (5)$$

$$D^- = \{(\alpha, \beta) \in D | p' \text{ cross } 0 \text{ before } p \text{ cross } 0\}, \quad (6)$$

$$D^+ = \{(\alpha, \beta) \in D | p \text{ cross } 0 \text{ before } p' \text{ cross } 0\}, \quad (7)$$

$$l = \{0\} \times (0, \bar{\beta}], \quad (8)$$

$$D_0 = D \setminus (l \cup D^- \cup D^+). \quad (9)$$

We have that D^-, D^+ are open in D , not empty and disjoint. By Lemma 2, we see that $l \cup D^-$ is open in D , so D_0 is compact. Note that for any $\beta \in [0, \bar{\beta}]$, there is $D_0 \cap ([0, \bar{\alpha}] \times \{\beta\}) \neq \emptyset$. And for $\beta = 0$, only $(0, 0) \in D_0$.

Also let's set

$$\mathbf{B} = \{p | (p, q) \text{ is a solution to (P) for } (\alpha, \beta) \in D_0\}. \quad (10)$$

We see that for any $p \in \mathbf{B}$, the properties in (6), (7) are not satisfied, and p, p' can not vanish at the same time by uniqueness of the solution. So we get $0 < p(t) \leq 1$, for $0 < t < \infty$, if $p \in \mathbf{B}$.

Now let's restrict (α, β) in D_0 .

We have already seen that in D_0 , the problem (P) has a unique solution (p, q) satisfying $0 < p \leq 1$, and by eq. (2) we see that q is bounded in any finite interval for $\beta > 0$. And for $\beta = 0, \alpha = 0$, there are $p = 1, q$ is expressed by $J_{3/2}(t)$ (see (12) for $p = 1$). Thus we get the following result.

Lemma 6 *For $(\alpha, \beta) \in D_0$, the problem (P) has a unique solution and for any finite $\bar{t} > 0$, $q(t)$ is uniformly bounded for $(t, \alpha, \beta) \in [0, \bar{t}] \times D_0$, and $0 < p \leq 1$ for $(t, \alpha, \beta) \in [0, \infty) \times D_0$.*

Next, we want to find two linearly independent solutions of the equation

$$Q'' + \frac{2}{t}Q' + \left(1 - \frac{2p^2}{t^2}\right)Q = 0. \quad (11)$$

Let

$$Q(t) = \frac{1}{\sqrt{t}}y(t),$$

which reduces the equation (11) to

$$y'' + \frac{1}{t}y' + \left(1 - \frac{\nu^2}{t^2} + \frac{2(1 - p^2(t))}{t^2}\right)y = 0, \quad (12)$$

where $\nu = 3/2$.

Consider the Bessel functions

$$J_{3/2}(t) = \left(\frac{2}{\pi t}\right)^{1/2} \left(\frac{\sin t}{t} - \cos t\right), \quad (13)$$

$$J_{-3/2}(t) = -\left(\frac{2}{\pi t}\right)^{1/2} \left(\frac{\cos t}{t} + \sin t\right). \quad (14)$$

Let t_0 be the first positive zero of $J_{3/2}(t)$. Choose $t_1 > t_0$, so that $J_{3/2}(0) = J_{3/2}(t_0) = 0, J_{3/2}(t) > 0$, for $t \in (0, t_0)$, and $J_{3/2}(t) < 0$, for $t \in (t_0, t_1)$.

Lemma 7 *For each $(\alpha, \beta) \in D_0, p \in \mathbf{B}$, there are two linearly independent solutions $y_p^{(1)}(t), y_p^{(2)}(t)$ to the equation (12) uniquely determined by the asymptotics*

$$y_p^{(1)}(t) \sim J_{3/2}(t) \sim \frac{1}{3} \left(\frac{2}{\pi}\right)^{1/2} t^{3/2}, \quad (15)$$

$$y_p^{(2)}(t) \sim J_{-3/2}(c_0 t) \sim -\left(\frac{2}{\pi}\right)^{1/2} (c_0 t)^{-3/2}, \quad (16)$$

as $t \rightarrow 0$, where

$$c_0 = \sqrt{1 + \frac{4\alpha}{\lambda\rho_0^2}}.$$

There is singularity only for $y_p^{(2)}$ at the origin.

Proof First of all we have the Wronskian

$$W(J_{3/2}, J_{-3/2}) = \frac{2}{\pi t}.$$

Define $y_p^{(1)}$ for $0 < t < \infty$ by the integral equation

$$\begin{aligned} y_p^{(1)}(t) = J_{3/2}(t) &+ \pi J_{-3/2}(t) \int_0^t \sigma J_{3/2}(\sigma) \frac{p^2(\sigma) - 1}{\sigma^2} y_p^{(1)}(\sigma) d\sigma \\ &- \pi J_{3/2}(t) \int_0^t \sigma J_{-3/2}(\sigma) \frac{p^2(\sigma) - 1}{\sigma^2} y_p^{(1)}(\sigma) d\sigma. \end{aligned}$$

By iteration method one can show that $y_p^{(1)}(t)$ is uniquely defined without singularity.

Now let's consider how to define $y_p^{(2)}(t)$. By eq. (6.2) it's not hard to see that

$$p(t) = 1 - \frac{\alpha}{\lambda\rho_0^2} t^2 + O(t^4),$$

as $t \rightarrow 0$. Let

$$\begin{aligned} s &= c_0 t, \\ y(t) &= z(s) \\ R(s) &= c_0^{-2} \left(\frac{4\alpha}{\lambda\rho_0^2} - \frac{2(1 - p^2(t))}{t^2} \right). \end{aligned}$$

By simple calculation we see that

$$R(s) = O(s^2),$$

as $s \rightarrow 0$. So now eq. (6.11) becomes

$$z'' + \frac{1}{s} z' + \left(1 - \frac{\nu^2}{s^2} - R(s) \right) z = 0.$$

Define $z(s)$ by

$$\begin{aligned} z(s) &= J_{-3/2}(s) + \frac{\pi}{2} J_{-3/2}(s) \int_0^s \sigma J_{3/2}(\sigma) R(\sigma) z(\sigma) d\sigma \\ &\quad - \frac{\pi}{2} J_{3/2}(s) \int_0^s \sigma J_{-3/2}(\sigma) R(\sigma) z(\sigma) d\sigma, \\ z(s) &\sim J_{-3/2}(s), \text{ as } s \rightarrow 0. \end{aligned}$$

Now let $y_p^{(2)}(t) = z(s)$, and then the lemma is done.

So now

$$Q_p^{(j)}(t) = \frac{1}{\sqrt{t}} y_p^{(j)}(t), j = 1, 2 \quad (17)$$

forms a basis of eq. (11) with the Wronskian

$$W(Q_p^{(1)}, Q_p^{(2)}) = \frac{\gamma}{t^2}, \quad (18)$$

where

$$\gamma = \lim_{t \rightarrow 0} t^2 W(Q_p^{(1)}, Q_p^{(2)}) = \frac{5}{3\pi} c_0^{-3/2}.$$

Lemma 8

$$m = \sup_{p \in \mathbf{B}} \inf_{t \in [0, t_1]} y_p^{(1)}(t) < 0, \quad (19)$$

and

$$m_0 = \sup_{p \in \mathbf{B}} \inf_{t \in [0, t_1]} Q_p^{(1)}(t) < 0. \quad (20)$$

Proof Because $J_{3/2}(0) = 0 = J_{3/2}(t_0)$, and for any $p \in \mathbf{B}, 0 < p \leq 1$, by Sturm comparison principle, there is a zero of $y_p^{(1)}(t)$ between 0 and t_0 . Since $y_p^{(1)}(t)$ is not a trivial solution, by the uniqueness theorem, $(y_p^{(1)}(t))', (y_p^{(1)}(t))''$ can be zero at the same time. Thus $y_p^{(1)}(t)$ must cross 0 in $(0, t_1)$ for any $p \in \mathbf{B}$, which implies

$$\inf_{t \in [0, t_1]} y_p^{(1)}(t) < 0. \quad (21)$$

So $m \leq 0$.

Suppose $m = 0$. Then there is a sequence $\{(\alpha^{(n)}, \beta^{(n)})\}_{n=1}^\infty \subset D_0$, such that

$$\inf_{t \in [0, t_1]} y_{p^{(n)}}^{(1)}(t) \rightarrow 0,$$

as $n \rightarrow \infty$, where $p^{(n)} = p(t, \alpha^{(n)}, \beta^{(n)})$. Because D_0 is compact, without loss of generality, assume

$$(\alpha^{(n)}, \beta^{(n)}) \rightarrow (\alpha^*, \beta^*) \in D_0,$$

as $n \rightarrow \infty$. By continuity, there is

$$\inf_{t \in [0, t_1]} y_{p^*}^{(1)}(t) = 0, \quad p^* = p(t, \alpha^*, \beta^*)$$

This is a contradiction because $(\alpha^*, \beta^*) \in D_0$, which implies p^* satisfies (21). Thus $m < 0$. And if $m_0 \geq 0$, then $m \geq 0$. So we also have $m_0 < 0$.

Lemma 9 *There is a small $\beta_1 > 0$, such that for any $\beta \in (0, \beta_1]$, if $\alpha \notin S_{\beta}^- \cup S_{\beta}^+$, then (A) is satisfied.*

Proof Suppose (p, q) is a solution to (2), (3) and (4), then by the variation of parameter, q satisfies the integral equation

$$q(t) = \mu Q_p^{(1)}(t) + \frac{\beta^2}{\gamma \rho_0^2} G(t), \quad 0 \leq t \leq t_1, \quad (22)$$

where

$$G(t) = Q_p^{(2)}(t) \int_0^t s^2 Q_p^{(1)}(s) q(s)^3 ds - Q_p^{(1)}(t) \int_0^t s^2 Q_p^{(2)}(s) q(s)^3 ds, \quad (23)$$

and

$$\mu = \frac{3}{\rho_0} \sqrt{\frac{\pi}{2\lambda}}.$$

By Lemma 6,7, we see that $G(t) = O(t^3)$, as $t \rightarrow 0$, and G is uniformly bounded in $[0, t_1] \times D_0$. Choose $\bar{\beta}_1 < \bar{\beta}$, so that if $0 < \beta < \bar{\beta}_1$, there is

$$\left| \frac{\beta^2}{\gamma \rho_0^2} G(t) \right| < \frac{\mu |m_0|}{2},$$

which implies by (22) and by Lemma 8

$$\inf_{t \in [0, t_1]} q(t) < \inf_{t \in [0, t_1]} \left(\mu Q_p^{(1)}(t) + \frac{\mu |m_0|}{2} \right) < \frac{\mu m_0}{2} < 0,$$

which means q crosses 0, since $q'(0) = 1$. And then q' or ρ' cross 0. Choose smaller positive $\beta_1 < \bar{\beta}_1$, such that $0 < \rho < \rho_0$ before ρ' crosses 0. So the lemma is proved.

7. The Solutions for Large β

Lemma 10 *There is a large $\beta_2 > 0$, such that if $\alpha \notin S_\beta^- \cup S_\beta^+$, then (B) is satisfied.*

Proof Recall the integral equation form we used in section 2,

$$\phi = -\alpha + \frac{1}{3} \int_{-\infty}^s (e^{2\sigma} - e^{-3s+5\sigma})(3\phi^2 + e^{2\sigma}\phi^3 + g_0^2 e^{2\sigma}\psi^2\phi) d\sigma, \quad (1)$$

$$\psi = \beta + \frac{1}{2} \int_{-\infty}^s (e^{2\sigma} - e^{-3s+5\sigma})(2(2\phi + e^{2\sigma}\phi^2)\psi + \lambda(e^{2\sigma}\psi^2 - \rho_0^2)\psi) d\sigma, \quad (2)$$

where

$$s = \log r, f(r) = 1 + e^{2s}\phi(s), \rho(r) = e^s\psi(s). \quad (3)$$

Suppose there is a sequence $\{(\alpha^{(n)}, \beta^{(n)})\}_{n=1}^\infty \subset (0, \infty) \times (0, \infty)$ with $\beta^{(n)} \rightarrow \infty$, as $n \rightarrow \infty$, and $\alpha^{(n)} \notin S_\beta^- \cup S_\beta^+$, such that (B) is not satisfied for each n . By Lemma 5 for each n , either (A) or (C) is satisfied. If (C) is satisfied, the theorem is proved. So we consider for each n , (A) is satisfied. Let us denote

$$\begin{aligned} f^{(n)} &= f(r, \alpha^{(n)}, \beta^{(n)}), \\ \rho^{(n)} &= \rho(r, \alpha^{(n)}, \beta^{(n)}). \end{aligned}$$

For each n , let $[0, r_n]$ be the maximal interval such that

$$|\rho^{(n)}| \leq \rho_0. \quad (4)$$

Let

$$\bar{r} = \inf_n(r_n).$$

We want to show $\bar{r} > 0$ (maybe ∞). If $\bar{r} = 0$, without loss of generality, assume $r_n \rightarrow 0$, as $n \rightarrow \infty$. Let

$$s_n = \log r_n, f^{(n)}(r) = 1 + e^{2s}\phi^{(n)}(s), \rho^{(n)}(r) = e^s\psi^{(n)}(s). \quad (5)$$

Then $\phi^{(n)}, \psi^{(n)}$ have uniform bounds in $(-\infty, s_n]$. But by (2), we have $\psi^{(n)}(s_n) \rightarrow +\infty$, as $n \rightarrow \infty$, which is a contradiction. So $\bar{r} > 0$.

Now choose $0 < \hat{r} < \infty$ so that (4) is satisfied for $r \in [0, \hat{r}]$ for all n . Then still by (2) we have that $\psi^{(n)}(s) \rightarrow +\infty$, as $n \rightarrow \infty$, for all $s \in (-\infty, \log(\hat{r})]$. This contradiction implies that the lemma is proved.

8. Proof of the Theorem

By Lemma 9 there is $\beta_1 > 0$ so that if $\alpha \notin S_{\beta_1}^- \cup S_{\beta_1}^+$, (A) is satisfied. By Lemma 10 there is $\beta_2 > \beta_1 > 0$ so that if $\alpha \notin S_{\beta_2}^- \cup S_{\beta_2}^+$, (B) is satisfied. For $\beta \in [\beta_1, \beta_2]$, by Lemma 2,3, there are $0 < \alpha_1 < \alpha_2$, so that if $\alpha \in (0, \alpha_1]$, then $\alpha \in S_{\beta}^-$, and if $\alpha \in [\alpha_2, \infty)$, then $\alpha \in S_{\beta}^+$. Define

$$I = [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]. \quad (1)$$

Let

$$S^- = \{(\alpha, \beta) \in I \mid f' \text{ cross } 0 \text{ before } f \text{ cross } 0\}, \quad (2)$$

$$S^+ = \{(\alpha, \beta) \in I \mid f \text{ cross } 0 \text{ before } f' \text{ cross } 0\}. \quad (3)$$

We have seen that S^-, S^+ are open, nonempty and disjoint.

By the topological lemma in [8](McLeod and Serrin), there is a continuum Γ connects $\beta = \beta_1$ and $\beta = \beta_2$. Define

$$\Omega_- = \{(\alpha, \beta) \in \Gamma \mid \text{(A) is satisfied}\}, \quad (4)$$

$$\Omega_+ = \{(\alpha, \beta) \in \Gamma \mid \text{(B) is satisfied}\}. \quad (5)$$

By the choice of β_1, β_2 , we see that Ω_-, Ω_+ are not empty and disjoint, and it's easy to show that Ω_-, Ω_+ are open in Γ . Thus there exists $(\alpha^*, \beta^*) \in \Gamma \setminus (\Omega_- \cup \Omega_+)$, such that (C) is satisfied. So $f(r, \alpha^*, \beta^*), \rho(r, \alpha^*, \beta^*)$ is a solution to the boundary value problem, and satisfying the properties

$$0 < f < 1, f' \leq 0, \quad (6)$$

$$0 < \rho < \rho_0, \rho' \geq 0, \quad (7)$$

for $0 < r < \infty$.

Finally we need to show $f' < 0, \rho' > 0$.

Suppose r_1 is a positive zero of f' . Then by eq. (1.10), there is

$$f'''(r_1) = 2 \left(\frac{1 - f^2}{r^3} + g_0^2 \rho \rho' \right) f > 0,$$

which is a contradiction. So $f' < 0$.

Now suppose r_2 is a zero point of ρ' . Then because $\rho' \geq 0$, there is $\rho''(r_2) = 0$. By eq. (1.11), we have

$$\rho'''(r_2) = 4 \left(\frac{f f'}{r^2} - \frac{f^2}{r^3} \right) \rho < 0,$$

which is a contradiction. So the theorem is proved.

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